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Generalized Fixed Point and Weak Convergence Theorems for New Nonlinear Mappings in Hilbert Spaces

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Abstract. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. In this article, we first deal with fundamental properties for generalized hybrid mappings in a Hilbert space. Then, we deal with fixed point theorems and weak convergence theorems for these nonlinear mappings in a Hilbert space.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is said to be *nonexpansive*, *nonspreading* [11], and *hybrid* [20] if

$$\|Tx - Ty\| \leq \|x - y\|,$$

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [3] and Goebel and Kirk [5]. From Baillon [2], and Takahashi and Yao [22], we know the following nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following:*

(i) T is nonexpansive;
(ii) T is nonspreading;
(iii) T is hybrid;
(iv) $2\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|Tx-y\|^2, \quad \forall x,y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to a fixed point of T.

Motivated by Theorem 1.1, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a class of nonlinear mappings called λ-hybrid in a Hilbert space. Kocourek, Takahashi and Yao [9] also introduced a more wide class of nonlinear mappings containing the class of λ-hybrid mappings: A mapping $T : C \to H$ is called *generalized hybrid* if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2$$

for all $x, y \in C$.

In this article, we first deal with fundamental properties for generalized hybrid mappings in a Hilbert space. Then, we deal with fixed point theorems and weak convergence theorems for these nonlinear mappings in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [19], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

From [13], a Hilbert space H satisfies Opial's condition, i.e., for a sequence $\{x_n\}$ of H such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2.3}$$

Let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x-y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [8]. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then, we have from (2.1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\
&= 0.
\end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1,1,1,\dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For a proof of existence of a Banach limit and its other elementary properties; see [16]. Using Banach limits, Takahashi and Yao [22] proved the following fixed point theorem.

Theorem 2.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C}\|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [19] for more details. We also know the following lemma.

Lemma 2.2 (Takahashi and Toyoda [21]). *Let F be a nonempty closed convex subset of a Hilbert space H, let P be the metric projection of H onto F and let $\{x_n\}$ be a sequence in H such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly.*

3 Nonlinear Mappings and Fixed Point Theorems

Let H be a Hilbert space. Let C be a nonempty closed convex subset of H and let $\lambda \in \mathbb{R}$. Then a mapping $T\colon C \to H$ is said to be λ-hybrid [1] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty\rangle \tag{3.1}$$

or equivalently

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\lambda\langle x - Tx, y - Ty\rangle \tag{3.2}$$

for all $x, y \in C$. A mapping $T : C \to H$ is called *generalized hybrid* [9] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.3}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$*-generalized hybrid* mapping. Hojo, Takahashi and Yao [6] proved the following result.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α and β be in $\mathbb{R}$. Then, a mapping $T : C \to H$ is $(\alpha,\ \beta)$-generalized hybrid if and only if it satisfies that*

$$\begin{aligned}\|Tx - Ty\|^2 \leq{} & (\alpha - \beta)\|x - y\|^2 \\ & + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2\end{aligned}$$

for all $x, y \in C$.

We can prove that a λ-hybrid mapping is generalized hybrid. In fact, suppose that T is λ-hybrid, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty\rangle \tag{3.4}$$

for all $x, y \in C$. Then, we have from (2.2) that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + (1-\lambda)(\|x - Ty\|^2 + \|Tx - y\|^2 - \|x - y\|^2 - \|Tx - Ty\|^2)$$

and hence $(2-\lambda)\|Tx - Ty\|^2 \leq \lambda\|x - y\|^2 + (1-\lambda)\|x - Ty\|^2 + (1-\lambda)\|Tx - y\|^2$. So, we have

$$(2-\lambda)\|Tx - Ty\|^2 + (\lambda - 1)\|x - Ty\|^2 \leq (1-\lambda)\|Tx - y\|^2 + \lambda\|x - y\|^2.$$

This implies that a λ-hybrid mapping is $(2-\lambda, 1-\lambda)$-generalized hybrid. Putting $x = Tx$ in (3.3), we have that for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|x - y\|^2 + (1-\beta)\|x - y\|^2$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that an $(\alpha,\ \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Using Theorem 2.1 and Banach limits, we can prove the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.2 (Kocourek, Takahashi and Yao [9]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers. According to Hojo, Takahashi and Yao [6], a mapping $U : C \to H$ is called (α, β, γ)-extended hybrid if

$$\begin{aligned}\alpha(1+\gamma)\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2\end{aligned}$$

for all $x, y \in C$. They proved the following theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is (α, β)-generalized hybrid if and only if $U : C \to H$ is (α, β, γ)-extended hybrid.*

Using Theorem 3.3, they proved the following fixed point theorem for generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β)-generalized hybrid mapping with $\alpha - \beta \geq 0$ of C into H. Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $1 \leq t \leq m$. Then, T has a fixed point in C.*

4 Weak Convergence Theorems

In this section, we first deal with a nonlinear ergodic theorem of Baillon's type [2] for generalized hybrid mappings in a Hilbert space. Before proving it, we need three lemmas. The first lemma is due to Takahashi, Yao and Kocourek [23].

Lemma 4.1 (Takahashi, Yao and Kocourek [23]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a generalized hybrid mapping. Then, $I - T$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ imply $z \in F(T)$.*

Using the technique developed by Takahashi [14], we can also prove the following lemma.

Lemma 4.2 (Hojo, Tahahashi and Yao [6]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Then, $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Aoyama, Iemoto, Kohsaka and Takahashi [1] proved the following lemma.

Lemma 4.3 (Aoyama, Iemoto, Kohsaka and Takahashi [1]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let $T: C \to C$ be a quasi-nonexpansive mapping, let $F(T)$ be the set of fixed points of T, let P be the metric projection of H onto $F(T)$, let $x \in C$,* *and let $\{S_n x\}$ be a sequence in C defined by*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

for $n \in \mathbb{N}$. If each weak cluster point of $\{S_n x\}$ belongs to $F(T)$, then $\{S_n x\}$ converges weakly to the strong limit of $\{PT^n x\}$.

Using Lemmas 4.1, 4.2 and 4.3, we can prove the following mean convergence theorem of Baillon's type [2] for generalized hybrid mappings in a Hilbert space.

Theorem 4.4 (Kocourek, Takahashi and Yao [9]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α and β be real numbers and let $T : C \to C$ be an (α, β)-generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^n x$.

Proof. Since $T : C \to C$ be an (α, β)-generalized hybrid mapping with $F(S) \neq \emptyset$, T is quasi-nonexpansive. Fix $x \in C$. Then, we have that for any $z \in F(T)$,

$$\|T^{n+1}x - z\| \leq \|T^n x - z\|$$

for all $n \in \mathbb{N}$. From Lemma 2.2, we have that $\{PT^n x\}$ converges strongly to an element $z \in F(T)$. Since $\{T^n x\}$ is bounded, $\{S_n x\}$ is bounded. So, there exists a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $S_{n_i} x \rightharpoonup v$. From Lemmas 4.1 and 4.2, we have $v \in F(T)$. So, we have from Lemma 4.3 that $\{S_n x\}$ converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^n x$. $\square$

Next, using Lemma 4.1, we can also prove a weak convergence theorem of Mann's type [12] generalized hybrid mappings in a Hilbert space.

Theorem 4.5 (Kocourek, Takahashi and Yao [9]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, \ldots.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} Px_n$.

Proof. Let $z \in F(T)$. Since T is quasi-nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&= \|x_n - z\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. So, we have that $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - z\|^2 \\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2.\end{aligned}$$

So, we have

$$\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $\|Tx_n - x_n\|^2 \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. By Lemma 4.1, we obtain $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|$ and $\lim_{n\to\infty} \|x_n - v_2\|$ exist. Suppose $v_1 \neq v_2$. Since H satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - v_1\| &= \lim_{i\to\infty} \|x_{n_i} - v_1\| \\ &< \lim_{i\to\infty} \|x_{n_i} - v_2\| \\ &= \lim_{n\to\infty} \|x_n - v_2\| \\ &= \lim_{j\to\infty} \|x_{n_j} - v_2\| \\ &< \lim_{j\to\infty} \|x_{n_j} - v_1\| \\ &= \lim_{n\to\infty} \|x_n - v_1\|.\end{aligned}$$

This is a contradiction. So, we have $v_1 = v_2$. This implies that $\{x_n\}$ converges weakly to some point v of $F(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.2 that $\{Px_n\}$ converges strongly to an element p of $F(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $-\|p - v\|^2 \geq 0$ and hence $p = v$. This means $v = \lim_{n\to\infty} Px_n$. This completes the proof. □

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